$A\vec{x} = \vec{b}$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

1.6 – More on Linear Systems and Invertible Matrices

$A\vec{y} = \vec{b}$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $A\vec{x} - A\vec{y} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Theorem 1.6.1: A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

pf: We know a system can have no solution or a unique solution. Suppose $A\vec{x} = \vec{b}$ has solutions $\vec{x}_1$ & $\vec{x}_2$. Then let $\vec{x}_0 = \vec{x}_1 - \vec{x}_2$. $A(\vec{x}_1 - \vec{x}_2) = A\vec{x}_1 - A\vec{x}_2$

$A\vec{x}_1 = A\begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ \vdots \\ x_{1m} \end{bmatrix}$ $A\vec{x} = A\begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2m} \end{bmatrix}$ $= \vec{b} - \vec{b} = \vec{0}$.

If k is any scalar, then $A(\vec{x}_1 + k\vec{x}_0) = A\vec{x}_1 + kA\vec{x}_0$

$= \vec{b} + \vec{0} = \vec{b}$

Infinitely many choices for k yield infinitely many solutions.

Theorem 1.6.2: If A is an invertible $n \times n$ matrix, then for every $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

Because A^{-1} is unique.

#4 Solve the system by inverting the coefficient matrix and using Theorem 1.6.2.

$$5x_1 + 3x_2 + 2x_3 = 4$$
$$3x_1 + 3x_2 + 2x_3 = 2$$
$$x_2 + x_3 = 5$$

Matrix equation:

$$\underbrace{\begin{bmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}}_{\vec{b}}$$

$$\left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{bmatrix}$$

so

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}}_{\vec{b}} = \begin{bmatrix} 1 \\ -11 \\ 16 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1 \\ -11 \\ 16 \end{bmatrix}$$

#11 Solve the linear systems (Using the given values for the b's) solve the systems together by reducing an appropriate augmented matrix to reduced row echelon form.

$$\begin{aligned} 4x_1 - 7x_2 &= \boxed{b_1} \\ x_1 + 2x_2 &= \boxed{b_2} \end{aligned}$$

i. $b_1 = 0,\ b_2 = 1$
ii. $b_1 = -4,\ b_2 = 6$
iii. $b_1 = -1,\ b_2 = 3$
iv. $b_1 = -5,\ b_2 = 1$

$$\left[\begin{array}{cc|cccc} 4 & -7 & 0 & -4 & -1 & -5 \\ 1 & 2 & 1 & 6 & 3 & 1 \end{array}\right]$$

follows the instructions strictly.

$$\left[\begin{array}{cc|c} 4 & -7 & b_1 \\ 1 & 2 & b_2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{2}{15}b_1 + \frac{7}{15}b_2 \\ 0 & 1 & -\frac{1}{15}b_1 + \frac{4}{15}b_2 \end{array}\right]$$

i) $(x_1, x_2) = \left(\frac{7}{15}, \frac{4}{15}\right)$

ii) $(x_1, x_2) = \left(-\frac{8}{15} + \frac{30}{15},\ \frac{4}{15} + \frac{24}{15}\right) = \left(\frac{22}{15}, \frac{28}{15}\right)$

etc.

since there are no restrictions on b_1, b_2, this system will always have a solution.

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 2 & b_1 \\ -2 & 1 & 5 & 1 & b_2 \\ -3 & 2 & 2 & -1 & b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right]$$

#17 Determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent.

$$\begin{array}{rrrrcl} x_1 & -x_2 & +x_3 & +2x_4 & = & b_1 \\ -2x_1 & +x_2 & +5x_3 & +x_4 & = & b_2 \\ -3x_1 & +2x_2 & +2x_3 & -x_4 & = & b_3 \\ 4x_1 & -3x_2 & +x_3 & +3x_4 & = & b_4 \end{array}$$

rref:

$$\left[\begin{array}{cccc|c} 1 & 0 & -8 & -3 & -b_1-b_2 \\ 0 & 1 & -11 & -5 & -2b_1-b_2 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_4 \end{array}\right]$$

This is consistent if $b_1-b_2+b_3=0$
and $-2b_1+b_2+b_4=0$

this becomes $\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -2 & -1 & 0 \end{array}\right]$

Which has a solution if

$b_1 = b_3+b_4$, $b_2 = 2b_3+b_4$

Theorem 1.6.4 Equivalent Statements (extends Theorem 1.5.3)
If A is an $n \times n$ matrix, then the following are equivalent.

a) A is invertible.
b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
c) The reduced row echelon form of A is I_n.
d) A is expressible as a product of elementary matrices.
New
e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

we already saw A, B invertible $\Rightarrow AB$ invertible

Theorem 1.6.5 Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

Pf: Let $\vec{x}_0$ be a solution of $B\vec{x} = \vec{0}$.

Then $(AB)\vec{x}_0 = A(B\vec{x}_0) = A\vec{0} = \vec{0}$

so $\vec{x}_0 = \vec{0}$ by thm 1.6.4 (a) and (b), applied to the invertible matrix AB.

Then $B\vec{x} = \vec{0}$ has only the trivial solution

$\Rightarrow$ B is invertible.

$A = A(BB^{-1}) = (AB)B^{-1}$.

Since AB and B^{-1} are invertible, so is their product. Thus A is invertible.